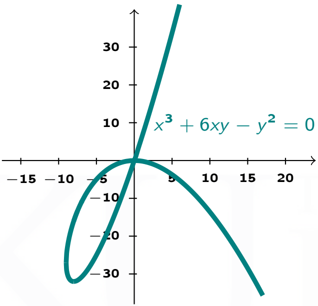
Section 3.8

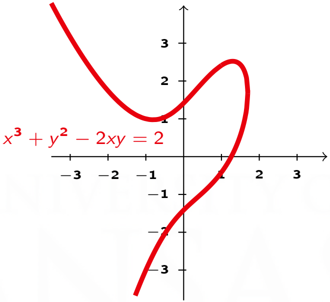
Implicit Differentiation

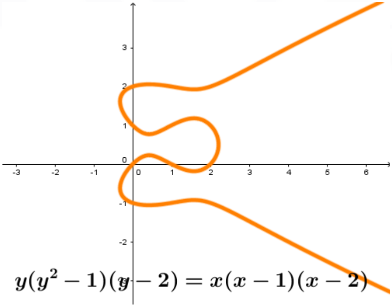
(1) Implicit Functions

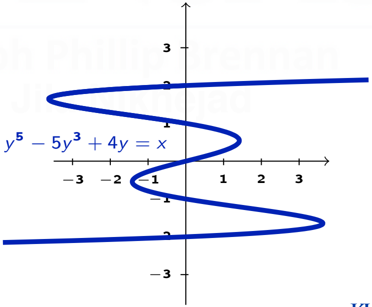
(2) Implicit Differentiation

(3) Derivatives of Logarithmic Functions


$x^3 + 6xy - y^2 = 0$
−15 −10 −5 5 10 15 20
30 20 10 −10 −20 −30


$x^3 + y^2 - 2xy = 2$
−3 −2 −1 1 2 3
3 2 1 −1 −2 −3

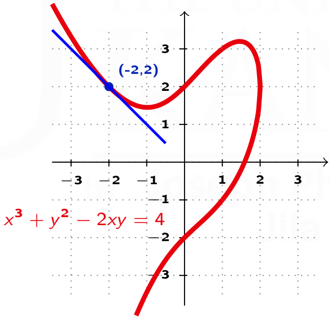

$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$
-3 -2 -1 0 1 2 3 4 5 6
3 2 1 0 -1 -2


$y^5 - 5y^3 + 4y = x$
−3 −2 −1 1 2 3
3 2 1 −1 −2 −3

Implicit Functions and Implicit Differentiation

In every equation that involves y and x, we can regard y as a function of x, except for where the derivative does not exist. Even if we cannot solve for y explicitly, it is still an implicit function of x.

$x^3+y^2-2xy = 4$



What is the slope of the tangent line at the point $(-2, 2)$?

Differentiating a function which is defined implicitly is called implicit differentiation, and is an application of the chain rule.

Idea: Differentiate both sides of the equation, then solve for $y'(x)$.

Implicit Differentiation

Example I(a): The circle of radius 5 is defined by the equation $x^2 + y^2 = 25$.
To find $y'(x)$, first differentiate both sides of the equation.

$$\frac{d}{dx}\left(x^2 + y(x)^2\right) = \frac{d}{dx}(25)$$

Note that $\frac{d}{dx}\left(y(x)^2\right) = 2y(x)y'(x)$, so this equation becomes

$$2x + 2y(x)y'(x) = 0.$$

Now abbreviate $y = y(x)$ and $y' = y'(x)$ and solve for $y'(x)$:

$$2x + 2yy' = 0 \qquad \therefore \qquad 2yy' = -2x \qquad \therefore \qquad \boxed{y' = -\frac{x}{y}}$$

Implicit Differentiation

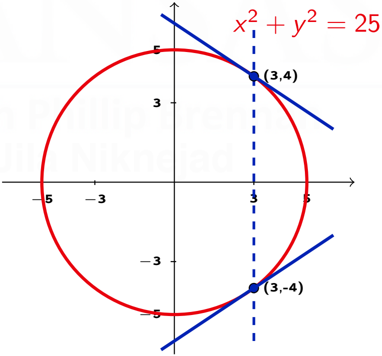
The equation $x^2 + y^2 = 25$ implicitly defines a function whose derivative with respect to x is

$$\frac{dy}{dx} = \frac{-x}{y}$$

Notice that the derivative depends upon more than just an x-value! Both an x and y value must be specified.

Example I(b): What is the slope of the tangent line to the circle at $x = 3$?

This question is ambiguous because there are two points on the circle at $(3, 4)$ and $(3, -4)$, whose tangent lines have different slopes: $-3/4$ and $3/4$.



Example II, Implicit Differentiation

By viewing y as an implicit function of x, we are viewing y as some function whose formula, $f(x)$, is unknown, but which we can differentiate.

Implicit differentiation is an application of the chain rule:

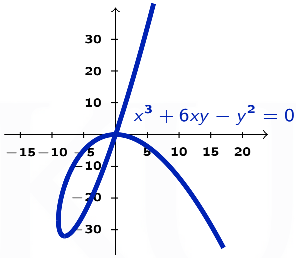
$$\frac{d}{dx}(y) = \frac{dy}{dx} \qquad \frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx} \qquad \frac{d}{dx}\left(e^y\right) = e^y\frac{dy}{dx}$$

The product and quotient rules still apply:

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y \qquad \frac{d}{dx}\left(x^2y^2\right) = (x^2)(2yy') + (2x)(y^2)$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y - xy'}{y^2} \qquad \frac{d}{dx}\left(\frac{x+1}{y+1}\right) = \frac{(y+1)(1) - (x+1)(y')}{(y+1)^2}$$

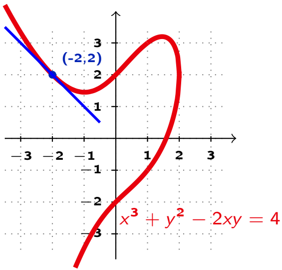
Example III



Example III(a): *Find y' if $x^3 + 6xy - y^2 = 0$.*

$$3x^2 + 6(xy' + y) - 2yy' = 0$$

$$\boxed{y' = \frac{-3x^2 - 6y}{6x - 2y}}$$



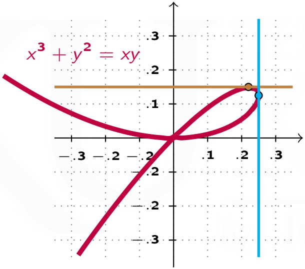
Example III(b): *Find the tangent line to the curve $x^3 + y^2 - 2xy = 4$ at $(-2, 2)$.*

$$y' = \frac{-3x^2 + 2y}{2y - 2x} \qquad y'\Big|_{x=-2,y=2} = -1$$

Answer: $\boxed{y - 2 = -1(x + 2) \text{ or } y = -x}$

Example III

Example III(c): *For the equation* $x^3 + y^2 = xy$ *find the points for which the tangent line is horizontal or vertical.*



Differentiate implicitly:

$$\frac{dy}{dx} = \frac{y - 3x^2}{2y - x}$$

Horizontal tangent line: $\frac{dy}{dx} = 0$ at $\left(\frac{2}{9},\ \frac{4}{27}\right)$

Vertical tangent line: $\frac{dy}{dx}$ undefined at $\left(\frac{1}{4}, \frac{1}{8}\right)$

Example IV, Comparing dy/dx and dx/dy

Differentiate the following equations with respect to each of x and y. Watch what happens!

(I) $9x^2 + xy + 9y^2 = 19$

$$\frac{dy}{dx} = \frac{-18x - y}{x + 18y} \qquad \frac{dx}{dy} = \frac{-18y - x}{18x + y}$$

(II) $\sqrt{x+y} = x^2y^2$

$$\frac{dy}{dx} = \frac{4xy^2\sqrt{x+y} - 1}{-1 + 4x^2y\sqrt{x+y}} \qquad \frac{dx}{dy} = \frac{-1 + 4x^2y\sqrt{x+y}}{4xy^2\sqrt{x+y} - 1}$$

(III) $e^{xy} = e^{4x} - e^{5y}$

$$\frac{dy}{dx} = \frac{4e^{4x} - ye^{xy}}{xe^{xy} + 5e^{5y}} \qquad \frac{dx}{dy} = \frac{xe^{xy} + 5e^{5y}}{4e^{4x} - ye^{xy}}$$

Derivatives of Logarithmic Functions

Remember that the logarithm function is defined by

$$y = \log_b(x) \qquad \Leftrightarrow \qquad x = b^y.$$

We can calculate the derivative of $y = \log_b(x)$ by implicitly differentiating the equation $x = b^y$ with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(b^y\right) \quad \Rightarrow \quad 1 = b^y \ln(b)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{b^y \ln(b)} = \frac{1}{x \ln(b)}$$

If $b = e$, then $\log_b(x) = \ln(x)$ and $\ln(b) = 1$.

Derivatives of Logarithmic Functions

$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x \ln(b)} \qquad\qquad \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$$

Example V

Derivatives of Logarithmic Functions

$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x\ln(b)} \qquad \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$$

(I) $y = \ln\left(\sqrt{x^2+1}\right)$ $\qquad \frac{dy}{dx} = \frac{x}{x^2+1}$

(II) $y = \ln\left(\ln\left(2x\right)\right)$ $\qquad \frac{dy}{dx} = \frac{1}{x\ln\left(2x\right)}$

(III) $y = \log_{10}\left(2+\sqrt{x}\right)$ $\qquad \frac{dy}{dx} = \frac{1}{2\ln(10)\sqrt{x}(2+\sqrt{x})}$

(IV) $y = \ln\left(\frac{x+1}{\sqrt{x-2}}\right)$ $\qquad \frac{dy}{dx} = \frac{x-5}{2(x+1)(x-2)}$

Example VI, Logarithms and Implicit Differentiation

Use implicit differentiation to find dy/dx if $\ln(xy) = x + y$.

$$\left(\frac{1}{xy}\right)\frac{d}{dx}(xy) = 1 + y'$$

$$\left(\frac{1}{xy}\right)(xy' + y) = 1 + y'$$

$$xy' + y = (1 + y')xy$$

$$xy' + y = xy + xyy'$$

$$xy' - xyy' = xy - y$$

$$\boxed{y' = \frac{xy - y}{x - xy}}$$

Example VII, Implicit Differentiation with Multiple Quantities

The area A and radius r of a circle are related by the well-known equation

$$A = \pi r^2.$$

Suppose that the radius is changing over time. *What can we say about the rate of change of the area?*

Solution: Both A and r are functions of time (t), so we can implicitly differentiate the equation with respect to t:

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) \qquad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

For example, if the radius is currently 10 cm and is decreasing at a rate of 4 cm/sec, then the instantaneous rate of change of area is

$$\frac{dA}{dt} = 2\pi(10 \text{ cm})(-4 \text{ cm/sec}) = -80\pi \text{ cm}^2/\text{sec}$$